

HOW STRONG ARE STREAK ARTIFACTS IN LIMITED ANGLE COMPUTED TOMOGRAPHY?

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ABSTRACT. In this article, we consider the limited angle problem in computed X-ray tomography. A common practice is to use the filtered back projection with the limited data. However, this practice has been shown to introduce the artifacts along some straight lines. In this work, we characterize the strength of these artifacts.

1. INTRODUCTION

Computed X-ray tomography (CT) is probably the best known imaging modality. The object of interest is scanned with X-rays and the loss of intensity along the rays provides the data for the image reconstruction. Roughly speaking, the obtained data is the Radon transform $\mathcal{R}f$ of the attenuation function f :

$$\mathcal{R}f(\theta, s) = \int_{\mathbb{R}} f(s\theta + t\theta^\perp) dt,$$

where $s \in \mathbb{R}$, $\theta \in \mathbb{S}^1$, and θ^\perp is the unit vector $\frac{\pi}{2}$ counterclockwise from θ . In order to reconstruct the image (i.e., the function f), one has to invert the Radon transform. This problem has attracted a significant amount of work in literature (see, e.g., [Nat01] and the reference therein).

Let us briefly introduce the well known filtered-backprojection formula to invert \mathcal{R}^{-1} . We recall the 1-dimensional Fourier transform \mathcal{F} and its inverse \mathcal{F}^{-1}

$$\begin{aligned} \mathcal{F}(g)(\tau) &= \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{-i\tau s} g(s) ds, \\ \mathcal{F}^{-1}(g)(s) &= \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{i\tau s} g(\tau) d\tau. \end{aligned}$$

Let $g(\theta, s) \in \mathcal{S}(\mathbb{S}^1 \times \mathbb{R})$. The one dimensional Lambda operator is

$$\Lambda_s g = \mathcal{F}_\tau^{-1}(|\tau| \mathcal{F}_s g).$$

We also define the back-projection operator

$$\mathcal{R}^*(g)(x) = \int_{\mathbb{S}^1} g(\theta, x \cdot \theta) d\theta.$$

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¹This work is motivated by [FQ13] and we borrow the basic notations from there.

Then, the well known filtered back-projection inversion formula for \mathcal{R} reads as (e.g., [Nat01])

$$(1) \quad f = \mathcal{B}\mathcal{R}f := \frac{1}{4\pi} \mathcal{R}^* \Lambda_s \mathcal{R}f.$$

One major disadvantage of the above reconstruction formula is that it is not local (due to the non-locality of the operator Λ_s). That is, in order to find f at a location x , one has to use the data at **all** angles θ and distances s . Lambda tomography has been proposed to overcome this advantage. Namely, let us consider the following Lambda reconstruction formula

$$(2) \quad \Lambda f := \mathcal{L}\mathcal{R}f := \frac{1}{4\pi} \mathcal{R}^* (-\partial_s^2) \mathcal{R}f.$$

This formula is local in the sense that in order to compute $\Lambda f(x)$, one only needs to use the data $\mathcal{R}f$ in an arbitrarily small neighborhood of the set

$$\{(\theta, s) : \theta \cdot x = s\}.$$

Although Λf is not equal to the image f , it is a good alternative for f , as follows. Let us recall the 2-dimensional Fourier transform and its inverse

$$\begin{aligned} \mathcal{F}(f)(\xi) &= \hat{f}(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-ix \cdot \xi} f(x) dx, \\ \mathcal{F}^{-1}(f)(x) &= \check{f}(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{ix \cdot \xi} f(\xi) d\xi. \end{aligned}$$

Then (see, e.g., [SK85])

$$(3) \quad \Lambda f = \mathcal{F}^{-1}(|\xi| \mathcal{F}(f)).$$

That is (see Section 2.4.2), Λ is a pseudo-differential operator of order one with the symbol

$$\sigma(x, \xi) \sim |\xi|, \text{ for large } |\xi|.$$

This, in particular, implies (see, e.g., [Hör83, Corollary 8.3.2])

$$WF(\Lambda f) = WF(f).$$

Here, $WF(f)$ and $WF(\Lambda f)$ are, respectively, the wave front set of f and Λf (see Definition 2.1). As explained in Section 2, the wave front set can be used to describe the singularities of a function (or, more generally, a distribution). Moreover, (3) implies that: $WF_s(f) = WF_{s-1}(\Lambda f)$ (see Section 2.2 for more discussion). That is, Lambda reconstruction emphasizes the singularities (image features) by one order. In some applications, such as edge detection, this is an advantage of Lambda reconstruction over the exact reconstruction. Therefore, one may concentrate on finding Λf (which is simpler to compute) instead to f . More discussion about Lambda tomography can be found in, e.g., [VKK81, SK85, FRS92, FFRS97]. The reader is also referred to [KLM95, RK96] for other kinds of local tomography.

In this article, we are interested in both exact construction formula (1) and the Lambda reconstruction formula (2).

Let us now turn our discussion to the main concern to this article, **the limited angle problem** (see, e.g., [KR92, RK92, KLM95, Kat97, FQ13, Kuc14]): $\mathcal{R}f$ is only known for $(s, \theta) \in \mathbb{R} \times \mathbb{S}_\Phi$, for some $0 < \Phi < \frac{\pi}{2}$. Here, $\mathbb{S}_\Phi \subsetneq \mathbb{S}^1$ is defined by:

$$\mathbb{S}_\Phi = \{\theta \in \mathbb{S}^1 : \theta = \pm(\cos \phi, \sin \phi), |\phi| < \Phi\}.$$

The reconstruction of f from the limited data problem is severely ill-posed. Instead of trying to reconstruct the exact value of f , a common practice is to reconstruct the **visible** singularities of f ; which are all the elements $(x, \xi) \in WF(f)$ such that $\frac{\xi}{|\xi|} \in \mathbb{S}_\Phi$. The reader is referred to [FQ13] for more discussion about the visible singularities.

Let us define the following limited angle version of \mathcal{R}^* :

$$\mathcal{R}_\Phi^*(g)(x) = \int_{\mathbb{S}_\Phi} g(\theta, x \cdot \theta) d\theta,$$

and limited angle version of \mathcal{B} and \mathcal{L} :

$$\begin{aligned} \mathcal{B}_\Phi g &= \frac{1}{4\pi} \mathcal{R}_\Phi^* \Lambda_s g, \\ \mathcal{L}_\Phi g &= \frac{1}{4\pi} \mathcal{R}_\Phi^* \left(-\frac{\partial^2}{\partial s^2} g \right). \end{aligned}$$

One can observe that \mathcal{B}_Φ (or \mathcal{L}_Φ) is equal to applying \mathcal{B} (respectively \mathcal{L}) to the limited data patched with zero outside the available range. It is shown in [FQ13, Kat97] that $\mathcal{B}_\Phi \mathcal{R}$ and $\mathcal{L}_\Phi \mathcal{R}$ reconstruct the visible singularities of f . However, they also create added singularities (artifacts) into the picture. These artifacts are generated from the singularities of f whose direction corresponds to the edges of \mathbb{S}_Φ (i.e., singularities along the direction $\mathbf{e}_1 = (\cos \Phi, \sin \Phi)$ or $\mathbf{e}_2 = (\cos \Phi, -\sin \Phi)$). Their locations line up along straight lines orthogonal to their direction (\mathbf{e}_1 or \mathbf{e}_2) and, hence, called streak artifacts (the reader is referred to [Kat97, FQ13] for detailed discussion).

In this article, we characterize the strength of above mentioned artifacts. In fact, we will analyze those generated by the general operators $\mathcal{B}_\Phi \mathcal{K} \mathcal{R}$ and $\mathcal{L}_\Phi \mathcal{K} \mathcal{R}$. Here, \mathcal{K} is the operator that multiplies by κ ,

$$\mathcal{K}g(\theta, s) = \kappa(\theta)g(\theta, s),$$

where

$$(4) \quad \kappa : \mathbb{S}^1 \rightarrow \mathbb{R} \text{ is a smooth even function such that } \kappa(\theta) > 0 \text{ for all } \theta \in \mathbb{S}_\Phi.$$

Obviously, if $\kappa \equiv 1$ then $\mathcal{B}_\Phi \mathcal{K} \mathcal{R} = \mathcal{B}_\Phi \mathcal{R}$ and $\mathcal{L}_\Phi \mathcal{K} \mathcal{R} = \mathcal{L}_\Phi \mathcal{R}$. Our results (see Theorem 3.1 and Section 3.2), in particular, show that when κ vanishes to order k at the end points $(\pm \cos \Phi, \pm \sin \Phi)$ of \mathbb{S}_Φ , then the artifacts are reduced by k orders.

It is worth mentioning that the same problem has been studied in [Kat97]. However, our approach and result are different from there. In particular, our result applies to general singularities, not only jumps.

The article is organized as follows. In Section 2, we introduce some basic concepts in microlocal analysis needed in this article. We then state the main result, Theorem 3.1, and its consequences in Section 3. Sections 4 and 5 are dedicated to the proof of Theorem 3.1.

2. BASIC NOTIONS IN MICROLOCAL ANALYSIS

In this section we introduce several concepts in microlocal analysis. We first discuss the definition of the wave front set and how to quantify it. We then provide some essential knowledge in pseudo-differential operators and Fourier integral operators, that is needed to understand the article.

Let $\Omega \subset \mathbb{R}^n$ be an open set. Throughout this article, we will denote by $\mathcal{D}'(\Omega)$ and $\mathcal{E}'(\Omega)$ the space of distributions and space of compactly supported distributions on Ω , respectively.

2.1. Wave front set. Here is the definition of the wave front set of a distribution:

Definition 2.1 (Wave Front Set [Hör71]). *Let $\Omega \subset \mathbb{R}^n$ be an open set, $u \in \mathcal{D}'(\Omega)$, and $(x_0, \xi_0) \in \mathbb{T}^*\Omega \setminus 0$ ². Then, u is microlocally smooth at (x_0, ξ_0) if there is a function $\varphi \in C_0^\infty(\Omega)$ satisfying $\varphi(x_0) \neq 0$ and an open cone V containing ξ_0 , such that $\mathcal{F}(\varphi f)$ is rapidly decreasing in V . That is, for any $N > 0$, there exists a constant C_N such that*

$$|\mathcal{F}(\varphi u)(\xi)| \leq C_N(1 + |\xi|)^{-N}, \text{ for all } \xi \in \mathbb{R}^n.$$

*The **wave front set** of u , denoted by $WF(u)$, is the complement of the set of all $(x_0, \xi_0) \in \mathbb{T}^*\Omega$ where u is microlocally smooth.*

An element $(x, \xi) \in WF(u)$ is called a singularity of u . The component x indicates the location of the singularity, while ξ indicates the direction of the singularity. For example, if u is the characteristic function of an open set $\mathcal{O} \Subset \Omega$ with the smooth boundary $\partial\mathcal{O}$, then $(x, \xi) \in WF(u)$ if and only if $x \in \partial\mathcal{O}$ and ξ is perpendicular to the tangent plane of $\partial\mathcal{O}$ at x . Detailed discussion can be found in [Pet83].

The study of the reconstruction of wave front set in limited data X-ray and related transforms was initiated in [Qui88, GU89, GU90a, GU90b, Qui93]. Similar study has become popular in many areas of imaging sciences.

In this article, to study the relationship between $WF(\mathcal{B}_\Phi \mathcal{KR}f)$, $WF(\mathcal{L}_\Phi \mathcal{KR}f)$ and $WF(f)$, we will intensively use the following rule for calculus of wave front set (see, [Hör71, Theorem 2.5.14]):

Theorem 2.2. *Let $\mathcal{T} : \mathcal{E}'(\mathbb{R}^2) \rightarrow \mathcal{D}'(\mathbb{R}^2)$ be a continuous linear operator whose Schwartz kernel μ satisfies $WF(\mu)' \subset (\mathbb{T}^*\mathbb{R}^2 \setminus 0) \times (\mathbb{T}^*\mathbb{R}^2 \setminus 0)$. Then for any $f \in \mathcal{E}'(\mathbb{R}^2)$,*

$$WF(\mathcal{T}) \subset WF(\mu)' \circ WF(f).$$

Here, $WF(\mu)'$ is the **twisted wave front set** of μ , defined by

$$WF(\mu)' = \{(x, \xi; y, -\eta) : (x, \xi; y, \eta) \in WF(\mu)\},$$

and, for any $A \subset \mathbb{T}^*\mathbb{R}^2 \times \mathbb{T}^*\mathbb{R}^2$,

$$A \circ WF(f) := \{(x, \xi) : (x, \xi; y, \eta) \in A, \text{ for some } (y, \eta) \in WF(f)\}.$$

² $\mathbb{T}^*\Omega \setminus 0$ is the cotangent bundle of Ω minus the zero section. It can be considered as $\Omega \times (\mathbb{R}^n \setminus 0)$.

2.2. Sobolev wave front set (singularities). An important issue in imaging sciences is how to quantify the strength of a singularity. The following definition can be used to serve that purpose:

Definition 2.3 (Sobolev Wave Front Set [Pet83]). *Let $\Omega \subset \mathbb{R}^n$ be an open set, $u \in \mathcal{D}'(\Omega)$, and $(x_0, \xi_0) \in \mathbb{T}^*\Omega \setminus 0$. Then u is in the space H^s microlocally at (x_0, ξ_0) if there is a function $\varphi \in C_0^\infty(\Omega)$ satisfying $\varphi(x_0) \neq 0$ and a function $p(\xi)$ homogeneous of degree zero and smooth on $\mathbb{R}^n \setminus 0$ with $p(\xi_0) \neq 0$, such that*

$$p(\xi) \mathcal{F}(\varphi u)(\xi) \in L^2(\Omega, (1 + |\xi|^2)^s).$$

The H^s -wave front set of u , denoted by $WF_s(u)$, is the complement of the set of all $(x_0, \xi_0) \in \mathbb{T}^\Omega$ where u is not microlocally in the space H^s .*

The notion of Sobolev wave front set has been used in imaging sciences to indicate the strength of singularities (see, e.g., [Qui93, QRS11, QR13]). We will use it to analyze the strength of the reconstructed singularities and artifacts generated by \mathcal{T}_m . The reader should keep in mind that, roughly speaking, the smaller s is, the **rougher** (i.e., **stronger**) a singularity $(x, \xi) \in WF_s(u)$ is. To compare two singularities $(x, \xi) \in WF(u_1)$ and $(y, \eta) \in WF(u_2)$ ³, one can make use of the following terminologies:

- i) (x, ξ) and (y, η) are of **the same order**, if for all $s \in \mathbb{R}$: $(x, \xi) \in WF_s(u_1)$ iff $(y, \eta) \in WF_s(u_2)$.
- ii) (x, ξ) is **stronger** than (y, η) , if there is $s \in \mathbb{R}$ such that $(x, \xi) \in WF_s(u_1)$ but $(y, \eta) \notin WF_s(u_2)$.

Since for any $u \in \mathcal{D}'(\Omega)$ (see, e.g., [Pet83])

$$\bigcup_{s \in \mathbb{R}} WF_s(u) = WF(u),$$

the above terminologies can be used to compare any two singularities.

2.3. Conormal singularities. In this section, we introduce a special kind of singularities, the conormal singularities, and how to quantify them. In many cases, concentrating on the conormal singularities can be beneficial for the analysis. In-depth discussion of conormal singularities and their applications in inverse scattering and wave propagation can be found in [GU90a, Jos98, Esw12, dUV12]). Its use in studying the artifacts of the x-ray transform in \mathbb{R}^3 with sources on a curve was suggested in [FLU03]. Our introduction below only touches the surface of the topic and is designed to serve our presentation in Section 3.

Assume that $S \subset \Omega$ be a smooth surface of co-dimension k . Let $h \in C^\infty(\Omega, \mathbb{R}^k)$ be a defining function for S with $\text{rank}(dh) = k$ on S . The class $I^r(S)$ consists of the distributions which locally can be written down as a finite sum of

$$u(x) = \int_{\mathbb{R}^k} e^{ih(x) \cdot \theta} a(x, \theta) d\theta,$$

where $a \in S^r(\Omega \times \mathbb{R}^k)$ (see Section 2.4.1 for the definition of $S^r(\Omega \times \mathbb{R}^k)$). We note that if $u \in I^r(S)$, then $WF(u) \subset N^*S$ (see, e.g., [Hör71]), where N^*S is the conormal bundle of S .

³The functions u_1, u_2 do not have to be defined on the same domain.

The order r is a good indication for the strength of the singularities of $u \in I^r(S)$. For example, assume that S is a smooth hypersurface in \mathbb{R}^n with non-vanishing Gaussian curvature. If u is a smooth density on S , then $u \in I^0(S)$ (see, [Ste93, Sections 5.7]). Meanwhile while, if u has Heaviside-type (i.e., jump) singularities at S , then $u \in I^{-1}(S)$ (see [Ste93, 6.14]).

In general, the **smaller** r is the **smoother** (i.e., **weaker**) the singularities of $u \in I^r(S)$ are.

In this article, we are only interested in the case S is a smooth curve in \mathbb{R}^2 . To fix our terminology, we introduce the following definition:

Definition 2.4. *Let $f \in \mathcal{D}'(\mathbb{R}^2)$. We say that $(x_0, \xi_0) \in WF(f)$ is a conormal singularity of order r along the curve S if there is $u \in I^r(S)$ such that*

$$(x_0, \xi_0) \notin WF(f - u).$$

One can use the order r to compare two conormal singularities $(x, \xi) \in WF(f_1)$ (along the curve S_1) and $(y, \eta) \in WF(f_2)$ (along the curve S_2), where f_1, f_2 are two distributions on \mathbb{R}^2 . For example, (x, ξ) is **weaker** than (y, η) , if there is $r \in \mathbb{R}$ such that (x, ξ) is of order r while (y, η) is not.

2.4. Pseudo-differential operators and Fourier integral operators. In this section, we make a brief introduction to pseudo-differential operators and Fourier integral operators. The reader is referred to [Hör71, Hör83, Trè80a, Trè80b, Dui11], among others, for a comprehensive introduction to the topics. The use of these operators to study the X-ray transform and its generalizations was introduced by Guillemin and Sternberg (see, e.g., [GS77, GS79]). It now becomes a standard technique in geometric integral transforms and tomography (see, e.g., [Qui88, GU89, GU90a, GU90b, Qui93, LQ00]).

One significant progress in microlocal analysis is theory of pseudo-differential operators with singular symbols, developed by Uhlmann, Guillemin, Melrose, and others (see, e.g., [MU79, GU81, AU85]). The use of that theory to analyze the X-ray transform, when the canonical relation is not a local canonical graph, was pioneered by Greenleaf and Uhlmann [GU89, GU90b]. It has been then exploited intensively to analyze other imaging scenarios (e.g., [FLU03, NC04, Fel05, FQ11, Esw12, FGN13, AFK⁺13]). Although we do not directly use that theory in this article, it strongly influences our analysis.

In the rest of this section, we do not attempt to provide the reader with an overview of pseudo-differential and Fourier integral operators. Instead, we only present the knowledge essential to understand our results.

2.4.1. The symbol classes. Let us start with the definition of the class $S^m(\Omega \times \mathbb{R}^N)$ of symbols (see, e.g., [Hör71]):

Definition 2.5. *Let $\Omega \subset \mathbb{R}^n$ be an open set. The space $S^m(\Omega \times \mathbb{R}^N)$ consists of all functions $a \in C^\infty(\Omega \times (\mathbb{R}^N \setminus 0))$ such that for any multi-indices α, β and $K \Subset \Omega$, there is a positive constant $C_{\alpha, \beta, K}$ such that*

$$(5) \quad |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha, \beta, K} (1 + |\xi|)^{m - |\alpha|}, \quad \text{for all } (x, \xi) \in K \times (\mathbb{R}^N \setminus 0).$$

The elements of $S^m(\Omega \times \mathbb{R}^N)$ are called symbols of order m .

We will also denote

$$S^{-\infty}(\Omega \times \mathbb{R}^N) = \bigcap_{m \in \mathbb{R}} S^m(\Omega \times \mathbb{R}^N).$$

Let $a(x, \xi)$ and $a'(x, \xi)$ be two symbols. We write $a \sim a'$ if $a - a' \in S^{-\infty}(\Omega \times \mathbb{R}^N)$.

If (5) is valid for $|\xi| \geq 1$, we say that $a \in S^m(\Omega \times \mathbb{R}^N)$ **for large** $|\xi|$. We also define $a \sim a'$ for large $|\xi|$, in the same way as above ⁴.

2.4.2. Pseudo-differential operators. Here is the definition of a pseudo-differential operator (see, e.g., [Hör83, Trè80a, Dui11]):

Definition 2.6. Let $a(x, y, \xi) \in S^m((\mathbb{R}^2 \times \mathbb{R}^2) \times \mathbb{R}^2)$. The operator $\mathcal{T} : \mathcal{E}'(\mathbb{R}^2) \rightarrow \mathcal{D}'(\mathbb{R}^2)$ defined by

$$\mathcal{T}f(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{i(x-y)\xi} a(x, y, \xi) f(y) dy d\xi$$

is called a pseudo-differential operator of order m with the amplitude $a(x, y, \xi)$.

The oscillatory integral on the right hand side might not converge in the normal sense, even when $f \in C_0^\infty(\mathbb{R}^2)$. The reader is referred to [Hör71, Hör83, Trè80a, Dui11] for its rigorous definition.

If $\mu(x, y)$ is the Schwartz kernel of the above operator \mathcal{T} , we write $\mu \in I^m(\Delta)$. Here, $\Delta \subset (\mathbb{T}^*\mathbb{R}^2 \setminus 0) \times (\mathbb{T}^*\mathbb{R}^2 \setminus 0)$ is the diagonal relation

$$\Delta = \{(x, \xi; x, \xi) : (x, \xi) \in \mathbb{T}^*\mathbb{R}^2 \setminus 0\}.$$

That is, $\mu \in I^m(\Delta)$ if

$$(6) \quad \mu(x, y) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i(x-y)\xi} a(x, y, \xi) d\xi,$$

where $a \in S^m((\mathbb{R}^2 \times \mathbb{R}^2) \times \mathbb{R}^2)$. The function $a(x, y, \xi)$ is also called the amplitude of μ .

If $\mu \in I^m(\Delta)$, then (e.g., [Hör71]):

$$(7) \quad WF(\mu)' \subset \Delta.$$

Consequently, due to Theorem 2.2, for any pseudo-differential operator \mathcal{T} [Hör71]:

$$(8) \quad WF(\mathcal{T}f) \subset WF(f).$$

That is, a pseudo-differential operator does not generate new singularities. In several references, i.e. [Trè80a, Pet83], the above inclusion is directly proved without explicitly employing Theorem 2.2.

If $a(x, y, \xi)$ has the form $a(x, y, \xi) = \sigma(x, \xi)$, then a is called the **symbol** of \mathcal{T} and μ . The reader is referred to, e.g., [Hör71, Trè80a] for the definition and formula of the symbol of a pseudo-differential operator in the general case. However, in this article, we only need to know the symbol in the aforementioned special case.

⁴We will occasionally drop the term “for large $|\xi|$ ” when it is clear from the context and not essential for the argument.

Lemma 2.7. *Let μ be defined by*

$$\mu(x, y) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i(x-y)\xi} a(x, \xi) d\xi,$$

where $a(x, \xi) \in S^m(\mathbb{R}^2 \times \mathbb{R}^2)$ for large $|\xi|$ such that $\partial_x^\alpha a(x, \xi)$ is locally integrable with respect to ξ for any orders α and $x \in \mathbb{R}^2$. Then, $\mu \in I^m(\Delta)$ with the symbol σ satisfying $\sigma \sim a$ for large $|\xi|$.

Proof. We can write $\mu = \mu_0 + \mu_1$, where

$$\mu_0(x, y) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i(x-y)\xi} c(|\xi|) a(x, \xi) d\xi,$$

and

$$\mu_1(x, y) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i(x-y)\xi} [1 - c(|\xi|)] a(x, \xi) d\xi.$$

Here, $c \in C^\infty(\mathbb{R})$ is such that $c(\tau) = 0$ for $|\tau| \leq 1$ and $c(\tau) = 1$ for $|\tau| \geq 2$.

Since $\partial_x^\alpha a(x, \xi)$ is locally integrable with respect to ξ for all $x \in \mathbb{R}^2$, we obtain $\mu_1 \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$. Therefore, see e.g. [Trè80a, Proposition 2.1], $\mu_1 \in I^m(\Delta)$ with the symbol $\sigma^1 \in S^{-\infty}(\mathbb{R}^2 \times \mathbb{R}^2)$.

On the other hand, $\mu_0 \in I^m(\Delta)$ with the symbol

$$\sigma^0(x, \xi) = c(|\xi|) a(x, \xi) \sim a(x, \xi), \quad \text{for large } |\xi|.$$

Therefore, $\mu \in I^m(\Delta)$ with the symbol σ satisfying $\sigma = \sigma^0 + \sigma^1 \sim a$ for large $|\xi|$. \square

Let $a(x, \xi) \in S^m(\mathbb{R}^2 \times \mathbb{R}^2)$. We say that $a(x, \xi)$ is elliptic near (x^*, ξ^*) if there is a conic neighborhood V of (x^*, ξ^*) and positive numbers C, ρ such that

$$|a(x, \xi)| \geq C(1 + |\xi|)^m, \quad \text{for all } (x, \xi) \in V, \text{ satisfying } |\xi| \geq \rho.$$

We also say that a pseudo-differential operator (of order m) \mathcal{T} is elliptic near (x^*, ξ^*) if its symbol is elliptic near (x^*, ξ^*) .

The following result tells us the effect of an elliptic operator on the singularity at (x^*, ξ^*) (see, e.g., [Trè80a, Pet83]):

Theorem 2.8. *Let $\mathcal{T} : \mathcal{E}'(\mathbb{R}^2) \rightarrow \mathcal{D}'(\mathbb{R}^2)$ be a pseudo-differential operator of order m . Assume that \mathcal{T} is elliptic near (x^*, ξ^*) . Then, for any $f \in \mathcal{E}'(\mathbb{R}^2)$ and $s \in \mathbb{R}$,*

$$(x^*, \xi^*) \in WF_s(f) \text{ if and only if } (x^*, \xi^*) \in WF_{s-m}(\mathcal{T}f).$$

The following technical term will be used in the statement of Theorem 3.1 a):

Definition 2.9. *Let $A \subset \Delta$ be a conic set that is open in the topology of Δ , induced from $(\mathbb{T}^*\mathbb{R}^2 \setminus 0) \times (\mathbb{T}^*\mathbb{R}^2 \setminus 0)$. We say that **near A , μ is microlocally in the space $I^m(\Delta)$ with the symbol σ** if the following holds: for each element $(x^*, \xi^*; x^*, \xi^*) \in A$ there exists $\mu_* \in I^m(\Delta)$ such that*

$$(x^*, \xi^*; x^*, \xi^*) \notin WF(\mu - \mu_*)',$$

and the symbol of μ_* is equal to $\sigma(x, \xi)$ in a conic neighborhood of (x^*, ξ^*) .

In Section 3.1, we will need the following more “microlocalized” version of Theorem 2.8:

Corollary 2.10. *Let $\mathcal{T} : \mathcal{E}'(\mathbb{R}^2) \rightarrow \mathcal{D}'(\mathbb{R}^2)$ be a linear operator whose Schwartz kernel $\mu \in \mathcal{D}'(\mathbb{R}^2 \times \mathbb{R}^2)$ satisfies $WF(\mu)' \subset (\mathbb{T}^*\mathbb{R}^2 \setminus 0) \times (\mathbb{T}^*\mathbb{R}^2 \setminus 0)$ and near $A \subset \Delta$ ⁵, μ is microlocally in $I^m(\Delta)$ with the symbol $\sigma(x, \xi)$. Assume that $(x^*, \xi^*; x^*, \xi^*) \in A$, $\sigma(x, \xi)$ is elliptic of order m near (x^*, ξ^*) , and*

$$(9) \quad \{(x^*, \xi^*; y, \eta) \in WF(\mu)' : (y, \eta) \in WF(f)\} \subset \Delta.$$

Then, for any $s \in \mathbb{R}$,

$$(x^*, \xi^*) \in WF_s(f) \text{ if and only if } (x^*, \xi^*) \in WF_{s-m}(\mathcal{T}f).$$

We provide its proof here to illuminate the use of the above assumptions.

Proof. Since $(x^*, \xi^*; x^*, \xi^*) \in A$ and μ is microlocally in $I^m(\Delta)$ near A with symbol σ , there is $\mu_* \in I^m(\Delta)$ such that

$$(10) \quad (x^*, \xi^*; x^*, \xi^*) \notin WF(\mu - \mu_*).$$

Moreover, the symbol of μ_* is $\sigma(x, \xi)$ near (x^*, ξ^*) . Let \mathcal{T}_* be the pseudo-differential operator of order m whose Schwartz kernel is μ_* . Then, \mathcal{T}_* is elliptic near (x^*, ξ^*) . Applying Theorem 2.8, we obtain, for any $s \in \mathbb{R}$,

$$(x^*, \xi^*) \in WF_s(f) \text{ if and only if } (x^*, \xi^*) \in WF_{s-m}(\mathcal{T}_*f).$$

Therefore, it suffices to prove that

$$(11) \quad (x^*, \xi^*) \notin WF(\mathcal{T}f - \mathcal{T}_*f).$$

Let us proceed to prove (11). We observe that

$$(12) \quad WF(\mu - \mu_*)' \subset WF(\mu)' \cup WF(\mu_*)' \subset WF(\mu)' \cup \Delta.$$

Here, to obtain the second inclusion, we have used (7) for $WF(\mu_*)'$.

The inclusion (12), together with $WF(\mu)' \subset (\mathbb{T}^*\mathbb{R}^2 \setminus 0) \times (\mathbb{T}^*\mathbb{R}^2 \setminus 0)$, implies

$$WF(\mu - \mu_*)' \subset (\mathbb{T}^*\mathbb{R}^2 \setminus 0) \times (\mathbb{T}^*\mathbb{R}^2 \setminus 0).$$

Hence, due Theorem 2.2,

$$(13) \quad WF(\mathcal{T}f - \mathcal{T}_*f) \subset WF(\mu - \mu_*)' \circ WF(f).$$

We now prove (11) by contradiction. To that end, let us assume

$$(x^*, \xi^*) \in WF(\mathcal{T}f - \mathcal{T}_*f).$$

Then, from (13), there is $(y, \eta) \in WF(f)$ such that

$$(14) \quad (x^*, \xi^*; y, \eta) \in WF(\mu - \mu_*)'.$$

From (12), we obtain

$$(x^*, \xi^*; y, \eta) \in WF(\mu)' \cup \Delta.$$

From (9), we arrive to

$$(x^*, \xi^*; y, \eta) \subset \Delta.$$

That is $(x^*, \xi^*; y, \eta) = (x^*, \xi^*; x^*, \xi^*)$. We, hence, arrive to a contradiction between (14) and (10). This finishes our proof. \square

⁵Here, A satisfies the condition in Definition 2.9.

2.4.3. Fourier integral operators (FIOs). In this section, we do not attempt to give the general definition and properties of FIOs. We, instead, introduce a special type of FIOs that is needed in this article. The interested reader is referred to [Hör71, Hör83, Trè80b, Dui11] for a comprehensive presentation on FIOs.

Let $\mathbf{e} \in \mathbb{R}^2$ be a unit vector and $a(x, y, \tau) \in S^{m+\frac{1}{2}}((\mathbb{R}^2 \times \mathbb{R}^2) \times \mathbb{R})$. Then, the operator $\mathcal{T} : \mathcal{E}'(\mathbb{R}^2) \rightarrow \mathcal{D}'(\mathbb{R}^2)$ defined by

$$(15) \quad \mathcal{T}(f)(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^2} \int_{\mathbb{R}} e^{i(x-y) \cdot \mathbf{e} \tau} a(x, y, \tau) f(y) d\tau dy$$

is a Fourier integral operator of order m with the phase function $\phi(x, y, \tau) = (x - y) \cdot \mathbf{e} \tau$ and amplitude function $a(x, y, \tau)$. The integral on the right hand side of (15) may not converge in the normal sense, even if $f \in C_0^\infty(\mathbb{R}^2)$. The reader is referred to, e.g., [Hör71, Trè80b] for its rigorous definition.

Remark 2.11. *One may notice the difference between the order of \mathcal{T} and that of the amplitude $a(x, y, \tau)$. This comes from the following general rule (see, e.g., [Hör71, Trè80b])*

$$(16) \quad \text{order of } \mathcal{T} = \text{order of } a + (N - n)/2.$$

Here, $n = n_x = n_y$ is the dimension of x and y , and N is the dimension of τ . In our case, $n = 2$ and $N = 1$.

Let $\mathcal{C} \subset (\mathbb{T}^*\mathbb{R}^2 \setminus 0) \times (\mathbb{T}^*\mathbb{R}^2 \setminus 0)$ be defined by

$$\mathcal{C} = \{(x, \gamma \mathbf{e}; x + t\mathbf{e}^\perp, \gamma \mathbf{e}) : x \in \mathbb{R}^2, \gamma, t \in \mathbb{R}, \gamma \neq 0\}.$$

Then, \mathcal{C} is called the canonical relation of \mathcal{T} ⁶. The following result tells us how \mathcal{T} transforms the wavefront set of a distribution $f \in \mathcal{E}'(\mathbb{R}^2)$ (see, e.g., [Hör71, Trè80b]):

$$(17) \quad WF(\mathcal{T}f) \subset \mathcal{C} \circ WF(f).$$

Let μ be the Schwartz kernel of \mathcal{T} . That is,

$$(18) \quad \mu(x, y) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}} e^{i(x-y) \cdot \mathbf{e} \tau} a(x, y, \tau) d\tau.$$

Then, we write $\mu \in I^m(\mathcal{C})$ (the interested reader is referred to [Hör71] for the definition of the Hörmander space $I^m(\mathcal{C})$ where \mathcal{C} is a general Lagrangian).

For any $\mu \in I^m(\mathcal{C})$, one has⁷

$$(19) \quad WF(\mu)' \subset \mathcal{C}.$$

Here, again, $WF(\mu)'$ is the twisted wave front set of μ , defined by

$$WF(\mu)' = \{(x, \xi; y, -\eta) : (x, \xi; y, \eta) \in WF(\mu)\}.$$

Assume that $a_{m+\frac{1}{2}}(x, y, \tau) \in C^\infty((\mathbb{R}^2 \times \mathbb{R}^2) \times (\mathbb{R} \setminus 0))$ is homogeneous of degree $m + \frac{1}{2}$ ⁸ and not identically zero on the projection $\mathcal{C}_{x,y}$ of \mathcal{C} on the (x, y) -space such that

$$a - a_{m+\frac{1}{2}} \in S^{m-\frac{1}{2}}((\mathbb{R}^2 \times \mathbb{R}^2) \times \mathbb{R}), \text{ for large } |\tau|.$$

⁶The reader is referred to [Hör71] for the canonical relation of a general FIO.

⁷The inclusion (19), in fact, implies (17), due to Theorem 2.2.

⁸That is, for all $s > 0$, $a_m(x, y, s\tau) = s^{m+\frac{1}{2}} a_m(x, y, \tau)$.

Then, we say that $\sigma(x, y, \tau) = a_{m+\frac{1}{2}}(x, y, \tau)|_{(x,y) \in \mathcal{C}_{x,y}}$ is the **principal symbol** of μ associated with the phase function $\phi(x, y, \tau) = (x-y) \cdot \mathbf{e} \tau$. The rigorous definition of the principal symbol of a general FIO is quite complicated and abstract. The interested reader is referred [Hör71] for the matter.

Lemma 2.12. *Let μ be defined as in (18), where $a(x, y, \tau) \in S^{m+\frac{1}{2}}((\mathbb{R}^2 \times \mathbb{R}^2) \times \mathbb{R})$ for large $|\tau|$ such that $\partial_x^\alpha \partial_y^\beta a(x, y, \tau)$ is locally integrable with respect to τ for all $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2$. Then, $\mu \in I^m(\mathcal{C})$ with the amplitude $a'(x, y, \tau) \sim a(x, y, \tau)$ for large $|\tau|$.*

The proof of Lemma 2.12 is similar to that of Lemma 2.7. We skip it for the sake of brevity.

The following result is helpful to analyze the strength of the artifacts in terms of their Sobolev order:

Theorem 2.13. *Let \mathcal{T} be defined in (15) and $f \in \mathcal{E}'(\mathbb{R}^2)$. Assume that $(x^*, \xi^* = \gamma_* \mathbf{e}) \in WF_s(\mathcal{T}f)$. Then, $(y^* = x^* + t_0 \mathbf{e}^\perp, \xi^*) \in WF_{s+m+\frac{1}{2}}(f)$ for some $t_0 \in \mathbb{R}$.*

Theorem 2.13 comes from (17) and the continuity of the FIOs between Sobolev spaces [Hör71, Theorem 4.3.2]. The reader should notice the order $(s + m + \frac{1}{2})$ (instead of $s + m$) in the conclusion. This is due to the fact that \mathcal{C} is not a local canonical graph⁹, and that order appears when applying [Hör71, Theorem 4.3.2].

The following technical term will be used in the statement of Theorem 3.1 b):

Definition 2.14. *Let $A \subset \mathcal{C}$ be a conic set that is open in the topology of \mathcal{C} , induced from $(\mathbb{T}^*\mathbb{R}^2 \setminus 0) \times (\mathbb{T}^*\mathbb{R}^2 \setminus 0)$. We say that **near A , μ is microlocally in the space $I^m(\mathcal{C})$ with the principal symbol $\sigma_0(x, y, \tau)$** if the following holds: for each element $(x^*, \xi^*; y^*, \eta^*) \in A$ there exists $\mu_* \in I^m(\mathcal{C})$ such that*

$$(x^*, \xi^*; y^*, \eta^*) \notin WF(\mu - \mu_*)'$$

and the principal symbol of μ_ is equal to $\sigma_0(x, y, \tau)$ for all (x, y) in a neighborhood of (x^*, y^*) .*

In Section 3.2, we will in fact need the following more “microlocalized” version of Theorem 2.13:

Corollary 2.15. *Let $\mathcal{T} : \mathcal{E}'(\mathbb{R}^2) \rightarrow \mathcal{D}'(\mathbb{R}^2)$ be a linear operator whose Schwartz kernel $\mu \in \mathcal{D}'(\mathbb{R}^2 \times \mathbb{R}^2)$ satisfies $WF(\mu) \subset (\mathbb{T}^*\mathbb{R}^2 \setminus 0) \times (\mathbb{T}^*\mathbb{R}^2 \setminus 0)$ and near $A \subset \mathcal{C}$ ¹⁰, μ is microlocally in $I^m(\mathcal{C})$. Assume that $(x^*, \xi^* = \gamma_* \mathbf{e}) \in WF_s(\mathcal{T}f)$ and*

$$\{(x^*, \xi^*; y, \eta) \in WF(\mu)' \cup \mathcal{C} : (y, \eta) \in WF(f)\} \text{ is a compact subset of } A.$$

Then, $(y^ = x^* + t_0 \mathbf{e}^\perp, \xi^*) \in WF_{s+m+\frac{1}{2}}(f)$ for some $t_0 \in \mathbb{R}$*

Corollary 2.15 can be proved in the same manner as Corollary 2.10, where Theorem 2.13 is used in place of Theorem 2.8. We skip it for the sake of brevity.

The following result is useful to analyze the artifacts when the original singularities are conormal:

Theorem 2.16. *Suppose that all the assumptions in Corollary 2.15 hold. Assume further that:*

⁹The reader is referred to [Hör71, Definition 4.1.5] for the definition of a local canonical graph.

¹⁰Here, A satisfies the condition in Definition 2.14.

- 1) *There are at most finitely many $y^* \in \mathbb{R}^2$ such that $y^* = x^* + te^\perp$ for some $t \in \mathbb{R}$ and $(y^*, \xi^*) \in WF(f)$.*
- 2) *For each such y^* , (y^*, ξ^*) is a conormal singularity of order r along a curve S which has nonzero curvature at y^* .*

Then, $(x^, \xi^* = \gamma_* \mathbf{e})$ is a conormal singularity of order at most $m + r$ along the line*

$$\ell = \{y \in \mathbb{R}^2 : y = x^* + te^\perp, t \in \mathbb{R}\}.$$

Theorem 2.16 can be proved in the same manner as Corollary 2.15, where Theorem 2.13 is replaced by [GU90a, Proposition 2.1]¹¹. We skip the details for the sake of brevity.

3. STATEMENT OF MAIN RESULT, ITS INTERPRETATION, AND ORGANIZATION OF THE PROOF

Let us denote by W_Φ the polar wedge

$$W_\Phi = \mathbb{R}_* \cdot \mathbb{S}_\Phi = \{r\theta : r \neq 0, \theta \in \mathbb{S}_\Phi\},$$

and χ_Φ the characteristic function of its closure $\text{cl}(W_\Phi)$.

Assume that m is a nonnegative real number and $\kappa \in C^\infty(\mathbb{S}^1)$ satisfies (4). Let \mathcal{T}_m be the linear operator whose Schwartz kernel is:

$$(20) \quad \mu_m(x, y) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i(x-y) \cdot \xi} |\xi|^m \kappa(\xi/|\xi|) \chi_\Phi(\xi) d\xi.$$

It can be easily shown that (see, e.g., [FQ13]): $\mathcal{T}_0 = \mathcal{B}_\Phi \mathcal{KR}$ and $\mathcal{T}_1 = \mathcal{L}_\Phi \mathcal{KR}$. Moreover, $\mathcal{T}_0 = \mathcal{B}_\Phi \mathcal{R}$ and $\mathcal{T}_1 = \mathcal{L}_\Phi \mathcal{R}$, if $\kappa \equiv 1$.

From now on, we will study the general operator \mathcal{T}_m for all real numbers $m \geq 0$. Our results will translate naturally to $\mathcal{B}_\Phi \mathcal{R}$, $\mathcal{L}_\Phi \mathcal{R}$, $\mathcal{B}_\Phi \mathcal{KR}$, and $\mathcal{L}_\Phi \mathcal{KR}$.

Assume that κ vanishes to **infinite** order at the boundary points of \mathbb{S}_Φ . Then

$$a(x, \xi) := |\xi|^m \kappa(\xi/|\xi|) \chi_\Phi(\xi)$$

is smooth on $\mathbb{R}^2 \times (\mathbb{R}^2 \setminus 0)$. Therefore, \mathcal{T} is a pseudo-differential operator with the symbol $\sigma(x, \xi) \sim a(x, \xi)$ for large $|\xi|$ (see Lemma 2.7). This, in particular, implies that $WF(\mathcal{T}_m f) \subset WF(f)$ (see Section 2.4.2). That is, \mathcal{T}_m does not generate artifacts. The reader is referred to [FQ13] for detailed arguments. We do not analyze this case any further in this article.

We now concentrate on the case κ only vanishes to **finite** order (or does not vanish at all, as in the case of $\mathcal{B}_\Phi \mathcal{R}$ and $\mathcal{L}_\Phi \mathcal{R}$) at the boundary points of \mathbb{S}_Φ . Then, $a(x, \xi)$ is no longer smooth on $\mathbb{R}^2 \times (\mathbb{R}^2 \setminus 0)$ with respect to the variable ξ . Therefore, \mathcal{T}_m is not a pseudo-differential operator in the standard sense. It is, instead, a pseudo-differential operator with a singular symbol. As we will show later in Theorem 3.1, the twisted wave front set of μ_m is contained in the union of three Lagrangians Δ , \mathcal{C}_1 , \mathcal{C}_2 . The part of the twisted wave front of μ_m in Δ , the diagonal, is responsible for the reconstruction of singularities. Those in $\mathcal{C}_1, \mathcal{C}_2$ are responsible for the generation of artifacts. By analyzing their strength (i.e.,

¹¹In this reference, the authors consider the mapping property of a class of pseudo-differential operators with the singular symbols between spaces of conormal distributions.

order), see Theorem 3.1 a)&b), we can describe the strength of the reconstructed singularities and artifacts, see Sections 3.1 and 3.2.

We now describe our results in details. Let $\Delta \subset (\mathbb{T}^*\mathbb{R}^2 \setminus 0) \times (\mathbb{T}^*\mathbb{R}^2 \setminus 0)$ be the diagonal relation

$$\Delta = \{(x, \xi; x, \xi) : (x, \xi) \in \mathbb{T}^*\mathbb{R}^2 \setminus 0\},$$

and

$$\Delta_\Phi = \{(x, \xi; x, \xi) \in \Delta : \xi \in \text{cl}(W_\Phi)\}.$$

For $j = 1, 2$, $\mathcal{C}_j \subset (\mathbb{T}^*\mathbb{R}^2 \setminus 0) \times (\mathbb{T}^*\mathbb{R}^2 \setminus 0)$ is defined by

$$\mathcal{C}_j = \{(x, \gamma \mathbf{e}_j; x + t \mathbf{e}_j^\perp, \gamma \mathbf{e}_j) : x \in \mathbb{R}^2, \gamma, t \in \mathbb{R}, \gamma \neq 0\}.$$

We recall that, as mentioned in the introduction,

$$\mathbf{e}_1 = (\cos \Phi, \sin \Phi) \text{ and } \mathbf{e}_2 = (\cos \Phi, -\sin \Phi).$$

Here is the main result of this article:

Theorem 3.1. *We have* ¹²

$$(21) \quad WF(\mu_m)' \subset \Delta_\Phi \cup \mathcal{C}_1 \cup \mathcal{C}_2.$$

Furthermore,

a) Near $\Delta \setminus (\mathcal{C}_1 \cup \mathcal{C}_2)$, μ_m is microlocally in the space $I^m(\Delta)$ with the symbol

$$\sigma(x, \xi) \sim |\xi|^m \kappa(\xi/|\xi|) \chi_\Phi(\xi), \quad \text{for large } |\xi|.$$

b) Let $\varkappa : [0, 2\pi] \rightarrow \mathbb{R}$ be defined by $\varkappa(\phi) = \kappa(\cos \phi, \sin \phi)$. Assume that \varkappa vanishes to order k at $\phi = \pm\Phi$ ¹³. Then, near $\mathcal{C}_j \setminus \Delta$, μ_m is microlocally in the space $I^{m-k-1/2}(\mathcal{C}_j)$. Moreover, given the phase function $\phi_j(x, y, \tau) = (x - y) \cdot \mathbf{e}_j \tau$, its principal symbol is

$$\sigma_0(x, y = x + t \mathbf{e}_j^\perp, \tau) = \frac{(-1)^j}{\sqrt{2\pi}} \frac{\varkappa^{(k)}((-1)^{j+1}\Phi)}{(i \operatorname{sgn}(\tau) t)^{k+1}} |\tau|^{m-k}.$$

Remark 3.2. We note that the order $(m - k - \frac{1}{2})$ of μ_m on $\mathcal{C}_j \setminus \Delta$, stated in Theorem 3.1 b), follows from the rule stated in Remark 2.11.

The reader is referred to Definitions 2.9 & 2.14 for the technical terms used in the statement of Theorem 3.1 a) & b). We now use Theorem 3.1 (together with Theorem 2.2, Corollary 2.10, Corollary 2.15, and Theorem 2.16) to analyze the reconstruction of singularities and generation of artifacts due to $\mathcal{T}_m f$. In the below discussion, we assume that $f \in \mathcal{E}'(\mathbb{R}^2)$.

Let us start with the inclusion (21). Applying Theorem 2.2, we obtain

$$WF(\mathcal{T}_m f) \subset WF(\mu_m)' \circ WF(f).$$

Therefore,

$$(22) \quad WF(\mathcal{T}_m f) \subset [\Delta_\Phi \circ WF(f)] \cup [\mathcal{C}_1 \circ WF(f)] \cup [\mathcal{C}_2 \circ WF(f)].$$

The first part on the right hand side is the set of all possible reconstructed singularities of f ; meanwhile, the other two parts contain all the possible artifacts. Let us now analyze them in more details.

¹²The reader is referred to Section 2.1 for the definition of the twisted wave front set $WF(\mu_m)'$.

¹³That is, $\varkappa^{(k)}(\pm\Phi) \neq 0$ and $\varkappa^{(l)}(\pm\Phi) = 0$, for all $0 \leq l \leq k-1$.

3.1. Reconstruction of singularities. From the inclusion (22), the set of possible reconstructed singularities is

$$\Delta_\Phi \circ WF(f) = \{(x, \xi) \in WF(f) : \xi \in \text{cl}(W_\Phi)\}.$$

Therefore, \mathcal{T}_m does not reconstruct $(x^*, \xi^*) \in WF(f)$ such that $\xi^* \notin \text{cl}(W_\Phi)$. Such a singularity (x^*, ξ^*) of f is called invisible.

On the other hand, assume that $(x^*, \xi^*) \in WF(f)$ and $\xi^* \in W_\Phi$, i.e., (x^*, ξ^*) is a **visible** singularity (see, e.g., [FQ13]). From Theorem 3.1 a) and Corollary 2.10,

$$(x^*, \xi^*) \in WF_{s-m}(\mathcal{T}_m f) \text{ if and only if } (x^*, \xi^*) \in WF_s(f).$$

That is, \mathcal{T}_m reconstructs the singularity of f at (x^*, ξ^*) ; and the reconstructed singularity is m order(s) stronger than the original singularity. In particular, the visible singularities are reconstructed with the same order if using $\mathcal{B}_\Phi \mathcal{R}$ and $\mathcal{B}_\Phi \mathcal{KR}$. Meanwhile, the visible singularities are emphasized by one order if using $\mathcal{L}_\Phi \mathcal{R}$ and $\mathcal{L}_\Phi \mathcal{KR}$ for the reconstruction.

3.2. Generation of artifacts. We notice that

$$\mathcal{C}_j \circ WF(f) = \{(x, \gamma \mathbf{e}_j) : (y = x + t\mathbf{e}_j^\perp, \gamma \mathbf{e}_j) \in WF(f), \text{ for some } t \in \mathbb{R}, \gamma \neq 0\}.$$

Assume that (x^*, ξ^*) is an artifact. That is, $(x^*, \xi^*) \notin WF(f)$ and $(x^*, \xi^*) \in WF(\mathcal{T}_m f)$. From (22), we obtain

$$(x^*, \xi^*) \in [(\mathcal{C}_1 \setminus \Delta) \circ WF(f)] \cup [(\mathcal{C}_2 \setminus \Delta) \circ WF(f)].$$

Therefore, there are $\gamma_* \neq 0$ and $j = 1$ or 2 such that $\xi^* = \gamma_* \mathbf{e}_j$. Moreover, there is at least one point $y^* \in \mathbb{R}^2$ such that

$$(23) \quad y^* = x^* + t\mathbf{e}_j^\perp, \text{ for some } t \neq 0, \text{ and } (y^*, \xi^* = \gamma_* \mathbf{e}_j) \in WF(f).$$

Each such $(y^*, \xi^*) \in WF(f)$ is called a singularity **corresponding to (or generating)** (x^*, ξ^*) . Due to Theorem 3.1 b) and Corollary 2.15, we obtain that if $(x, \xi) \in WF_s(\mathcal{T}_m f)$ then at least one of its corresponding singularity (y^*, ξ^*) satisfies $(y^*, \xi^*) \in WF_{s+(m-k)}(f)$. That is, using the Sobolev order to indicate the strength, we conclude:

- S.1) The artifacts are **at most** $(m - k)$ order(s) stronger than their strongest generating singularities if $m > k$.
- S.2) The artifacts are **at most** as strong as their strongest generating singularities if $m = k$.
- S.3) The artifacts are **at least** $(k - m)$ order(s) smoother than their strongest generating singularities if $k > m$.

Let us assume further that:

- A.1) (x^*, ξ^*) has only finitely many generating singularities $(y^*, \xi^*) \in WF(f)$, and
- A.2) each such generating singularity is conormal of order at most r along a curve S which has nonzero curvature at y^* .

Then, due to Theorem 2.16, (x^*, ξ^*) is a conormal singularity of order at most $(r + m - k - \frac{1}{2})$ along the curve

$$\ell = \{y : y = x^* + t\mathbf{e}^\perp, t \in \mathbb{R}\}.$$

That is, using the order of conormal singularity to indicate the strength, we conclude for such artifacts (x^*, ξ^*) :

- S.1') The artifacts are **at most** $(m - k - \frac{1}{2})$ order(s) stronger than their strongest generating singularities if $m > k + \frac{1}{2}$.
- S.2') The artifacts are **at most** as strong as their strongest generating singularities if $m = k + \frac{1}{2}$,
- S.3') The artifacts are **at least** $(k + \frac{1}{2} - m)$ order(s) smoother than their strongest generating singularities if $m < k + \frac{1}{2}$.

The descriptions in (S.1-3) and (S1'-3') can be easily interpreted for $\mathcal{B}_\Phi \mathcal{R}$, $\mathcal{L}_\Phi \mathcal{R}$, $\mathcal{B}_\Phi \mathcal{KR}$, and $\mathcal{L}_\Phi \mathcal{KR}$, by plugging the corresponding values of m and k .

In particular, for the case of $\mathcal{B}_\Phi \mathcal{R}$ (i.e., $m = k = 0$), using (S.2), we obtain that the artifacts are at most as strong as their strongest generating singularities. If we assume that (A.1) and (A.2) hold, then due to (S.3'), the artifacts are (at least) half an order **weaker** than the strongest generating singularities. This fact can be observed in [FQ13, Figure 1], where the artifacts are indeed visually weaker than the generating singularities.

3.3. Main ideas and structure of the proof of Theorem 3.1. Let us briefly discuss the main ideas of the proof of Theorem 3.1. The proof of (21) follows from the approach in [FQ13], which uses the relationship between the wave front set of a homogeneous distribution and that of its Fourier transform [Hör83, Theorem 8.1.4]. To prove parts a) and b), we use a partition of unity to decompose the integral (20) into five parts. The first one is an oscillatory integral with smooth amplitude and can be analyzed using the standard theory of pseudo-differential operator. Each of the other four integrals concentrates on a part (a ray) of ∂W_Φ . They can be analyzed using the common model introduced in Section 4.

We will proceed the proof of Theorem 3.1 as follows. In Section 4, we introduce a family of model oscillatory integrals. We show that their twisted wavefront set belongs to the union of two intersecting Lagrangians, one is the diagonal. We also proceed to compute their principal or full symbol on these Lagrangians. In Section 5, we present the proof of Theorems 3.1, using our understanding in Section 4.

Remark 3.3. *In our subsequent papers [Ngu14a, Ngu14b], we adapt the technique developed in this article to study the artifacts in limited data problem of spherical mean transform, which arises in several imaging modalities (such as thermo/photo-acoustic tomography, ultrasound tomography).*

4. MODEL OSCILLATORY INTEGRALS

Let m be a nonnegative real number and $\rho \in C^\infty(\mathbb{R}^2 \setminus 0)$ be homogeneous of degree zero¹⁴ and $\rho(\xi_1, \cdot)$ is compactly supported for any $\xi_1 \in \mathbb{R}$. We consider the oscillatory integral

$$(24) \quad \mu_\pm(x, y) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i(x-y) \cdot \xi} |\xi|^m \rho(\xi) \mathbb{H}(\pm \xi_2) d\xi.$$

Here, \mathbb{H} is the Heaviside function, defined by

$$\mathbb{H}(s) = \begin{cases} 1, & s \geq 0, \\ 0, & s < 0. \end{cases}$$

¹⁴A function $\rho \in C^\infty(\mathbb{R}^2 \setminus 0)$ is homogeneous of degree γ if for all $\tau > 0$, $\rho(\tau\xi) = \tau^\gamma \rho(\xi)$.

We also recall the diagonal canonical relation in $(\mathbb{T}^*\mathbb{R}^2 \setminus 0) \times (\mathbb{T}^*\mathbb{R}^2 \setminus 0)$

$$\Delta = \{(x, \xi; x, \xi) : (x, \xi) \in \mathbb{T}^*\mathbb{R}^2 \setminus 0\},$$

and define $\mathcal{C} \subset (\mathbb{T}^*\mathbb{R}^2 \setminus 0) \times (\mathbb{T}^*\mathbb{R}^2 \setminus 0)$ by

$$\mathcal{C} = \{(x, \xi; y, \xi) \in (\mathbb{T}^*\mathbb{R}^2 \setminus 0) \times (\mathbb{T}^*\mathbb{R}^2 \setminus 0) : x_1 - y_1 = 0, \xi_2 = 0\}.$$

Proposition 4.1. *We have*

$$(25) \quad WF(\mu_{\pm})' \subset \Delta \cup \mathcal{C}.$$

Furthermore,

a) Near $\Delta \setminus \mathcal{C}$, μ_{\pm} is microlocally in $I^m(\Delta)$ with the full symbol¹⁵:

$$\sigma(x, \xi) \sim |\xi|^m \rho(\xi) \mathbb{H}(\pm \xi_2).$$

b) Assume that $\rho(\xi)$ vanishes to order k at $\xi_2 = 0$ for any fixed $\xi_1 \neq 0$. Then, near $\mathcal{C} \setminus \Delta$, μ_{\pm} is microlocally in the space $I^{m-k-\frac{1}{2}}(\mathcal{C})$ with the principal symbol

$$(26) \quad \sigma_0(x, y, \xi_1) = \begin{cases} \frac{\pm 1}{\sqrt{2\pi}} \frac{1}{[i(y_2 - x_2)]^{k+1}} \varphi_{\pm}^{(k)}(0) \xi_1^{m-k}, & \xi_1 > 0, \\ \frac{\pm 1}{\sqrt{2\pi}} \frac{1}{[i(y_2 - x_2)]^{k+1}} \varphi_{\pm}^{(k)}(0) |\xi_1|^{m-k}, & \xi_1 < 0, \end{cases}$$

given the phase function $\phi(x, y, \xi_1) = (x_1 - y_1)\xi_1$. Here, $\varphi_{\pm} \in C^{\infty}(\mathbb{R})$ is defined by the formula

$$(27) \quad \varphi_{\pm}\left(\frac{\xi_2}{|\xi_1|}\right) = \rho(\xi), \quad \text{for all } \pm \xi_1 > 0 \text{ and } \xi_2 \in \mathbb{R}.$$

Before proving the Proposition 4.1, we would like to point out that the function $\varphi_{\pm} \in C^{\infty}(\mathbb{R})$ in (27) is well-defined since ρ is homogeneous of degree zero. Moreover, since ρ vanishes to order k at $\xi_2 = 0$, $\varphi_{\pm}(\tau)$ also vanish to order k at $\tau = 0$.

Proof. We only need to prove the proposition for μ_{+} . The proof for μ_{-} is similar.

Proof for (25). We follow the approach in [FQ13]. Let k be defined by

$$(28) \quad k(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ix \cdot \xi} |\xi|^m \rho(\xi) \mathbb{H}(\xi_2) d\xi.$$

Then,

$$\mu_{+}(x, y) = k(x - y).$$

Therefore, see [Hör83, page 270],

$$(29) \quad WF(\mu_{+}) \subset \{(x, \xi; y, -\xi) : (x - y, \xi) \in WF(k)\}.$$

Due to (28), up to a constant multiple, the Fourier transform of k is

$$K(\xi) = |\xi|^m \rho(\xi) \mathbb{H}(\xi_2),$$

which is a homogenous distribution with wave front set

$$WF(K) \subset \{(\xi, x) : \xi_2 = 0, x_1 = 0\} \cup \{(0, x) : x \in \mathbb{R}^2\}.$$

¹⁵From now on, we will drop the term “for large $|\xi|$ ” for the sake of convenience.

We recall the following rule for wave front set of homogeneous distribution (see [Hör83, Theorem 8.1.4]):

$$\begin{aligned} (x, \xi) \in WF(k) &\iff (\xi, -x) \in WF(K), \quad \text{if } \xi \neq 0 \text{ and } x \neq 0, \\ (0, \xi) \in WF(k) &\iff \xi \in \text{supp}(K), \quad \text{if } \xi \neq 0. \end{aligned}$$

Therefore,

$$(30) \quad WF(k) \subset \{(x, \xi) : x_1 = 0, \xi_2 = 0\} \cup \{(0, \xi) : \xi \in \text{supp}(K)\}.$$

Combining (29) and (30), arrive to

$$\begin{aligned} WF(\mu_+) \subset \{(x, \xi; y, -\xi) : x_1 - y_1 = 0, \xi_2 = 0\} \\ \cup \{(x, \xi; y, -\xi) : x - y = 0, \xi \in \text{supp}(K)\}. \end{aligned}$$

That is,

$$(31) \quad WF(\mu)' \subset \mathcal{C} \cup \{(x, \xi; x, \xi) : \xi \in \text{supp}(K)\}.$$

In particular, this implies

$$WF(\mu)' \subset \mathcal{C} \cup \Delta.$$

We have finished the proof for (25).

Proof for a). Let $(x^*, \xi^*; x^*, \xi^*) \in \Delta \setminus \mathcal{C}$, then $\xi_2^* \neq 0$. Let $\mathfrak{C} \in C^\infty(\mathbb{R}^2 \setminus 0)$ be homogeneous of degree zero such that $\mathfrak{C}(\xi) = 1$ in an open cone V_0 containing ξ^* and $\mathfrak{C}(\xi) = 0$ in a conic neighborhood of the set $\{\xi : \xi_2 = 0, \xi_1 \neq 0\}$. We define

$$\mu_*(x, y) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i(x-y) \cdot \xi} |\xi|^m \rho(\xi) \mathfrak{C}(\xi) \mathbb{H}(\xi_2) d\xi.$$

We observe that the function

$$b(x, \xi) = |\xi|^m \rho(\xi) \mathfrak{C}(\xi) \mathbb{H}(\xi_2)$$

satisfies $b(x, \xi) \in S^m(\mathbb{R}^2 \times \mathbb{R}^2)$ for large $|\xi|$ and $\partial_x^\alpha b(x, \xi)$ is locally integrable with respect to ξ for any multi-index α and any $x \in \mathbb{R}^2$. Therefore, $\mu_* \in I^m(\Delta)$ with the symbol $\sigma^* \sim b$ (see Lemma 2.7). In particular, in the conic neighborhood $\mathbb{R}^2 \times V_0$ of (x^*, ξ^*) ,

$$\sigma^*(x, \xi) \sim |\xi|^m \rho(\xi) \mathbb{H}(\xi_2).$$

It, therefore, suffices to prove (see Definition 2.9):

$$(x^*, \xi^*; x^*, \xi^*) \notin WF(\mu_+ - \mu_*).'$$

Indeed, similarly to (31), we obtain

$$WF(\mu - \mu_*)' \subset \mathcal{C} \cup \{(x, \xi; x, \xi) : \xi \in \text{supp}(K_0)\}.$$

Here,

$$K_0(\xi) = |\xi|^m [1 - \mathfrak{C}(\xi)] \rho(\xi) \mathbb{H}(\xi_2)$$

is, up to a constant multiple, the Fourier transform of $(\mu - \mu_*)$.

Since $\xi_2^* \neq 0$, one easily sees $(x^*, \xi^*; y^*, \xi^*) \notin \mathcal{C}$. Moreover, since $1 - \mathfrak{C}(\xi) = 0$ in a neighborhood V_0 of ξ^* , we have

$$(x^*, \xi^*; x^*, \xi^*) \notin \{(x, \xi; x, \xi) : \xi \in \text{supp}(K_0)\}.$$

Therefore,

$$(x^*, \xi^*; x^*, \xi^*) \notin WF(\mu_+ - \mu_*)'.$$

This finishes the proof for a).

Proof for b). We now analyze μ on $\mathcal{C} \setminus \Delta$. Let $(x^*, \xi^*; y^*, \xi^*) \in \mathcal{C} \setminus \Delta$, then $x_2^* \neq y_2^*$. Let $\mathcal{O} \subset \mathbb{R}^2 \times \mathbb{R}^2$ be an open set containing (x^*, y^*) such that $x_2 \neq y_2$ for any $(x, y) \in \mathcal{O}$. It suffices to prove that $\mu_+|_{\mathcal{O}}$ is in $I^{m-k-\frac{1}{2}}(\mathcal{C})$ with the stated symbol (see Definition 2.14).

For $(x, y) \in \mathcal{O}$, let us write

$$(32) \quad \mu_+(x, y) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}} e^{i(x_1-y_1)\xi_1} a(x, y, \xi_1) d\xi_1,$$

where

$$(33) \quad a(x, y, \xi_1) = \int_{\mathbb{R}} e^{i(x_2-y_2)\xi_2} |\xi|^m \rho(\xi) \mathbb{H}(\xi_2) d\xi_2.$$

Since $\rho(\xi_1, \cdot)$ is compactly supported, the integral on the right hand side of (33) is, in fact, over a finite interval. Therefore, one can easily see that $a \in C^\infty((\mathbb{R}^2 \times \mathbb{R}^2) \times (\mathbb{R} \setminus 0))$. We can also observe that $\partial_x^\alpha \partial_y^\beta a(x, y, \xi_1)$ is locally integrable with respect to ξ_1 for any $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2$ and any multi-indices α, β . Due to Lemma 2.12¹⁶, it suffices to show that $a(x, y, \xi_1) \in S^{m-k}(\mathcal{O} \times \mathbb{R})$ for large $|\xi_1|$ and its leading term is

$$(34) \quad a_{m-k}(x, y, \xi_1) = \begin{cases} \frac{1}{[i(y_2-x_2)]^{k+1}} \varphi_+^{(k)}(0) \xi_1^{m-k}, & \xi_1 > 0, \\ \frac{1}{[i(y_2-x_2)]^{k+1}} \varphi_-^{(k)}(0) |\xi_1|^{m-k}, & \xi_1 < 0, \end{cases}$$

Indeed, let us consider $\xi_1 > 0$. Using the change of variable $\xi_2 = \xi_1 \tau$, we obtain

$$a(x, y, \xi_1) = \xi_1^{m+1} \int_0^\infty e^{i(x_2-y_2)\xi_1\tau} (1+\tau^2)^{m/2} \rho(\xi_1, \tau\xi_1) d\tau.$$

Recalling that $\varphi_+ : \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$\varphi_+\left(\frac{\xi_2}{\xi_1}\right) = \rho(\xi), \quad \text{for all } \xi_1 > 0 \text{ and } \xi_2 \in \mathbb{R},$$

we arrive to

$$a(x, y, \xi_1) = \xi_1^{m+1} \int_0^\infty e^{i(x_2-y_2)\xi_1\tau} \psi_+(\tau) d\tau, \quad \text{for all } \xi_1 > 0,$$

where $\psi_+(\tau) = (1+\tau^2)^{m/2} \varphi_+(\tau)$. Since $x_2 \neq y_2$ for all $(x, y) \in \mathcal{O}$, we can write:

$$a(x, y, \xi_1) = \frac{1}{i(x_2-y_2)} \xi_1^m \int_0^\infty (e^{i(x_2-y_2)\xi_1\tau})_\tau \psi_+(\tau) d\tau.$$

Taking integration by parts and noticing that ψ_+ is compactly supported, we obtain

$$a(x, y, \xi_1) = \frac{1}{i(y_2-x_2)} \xi_1^m \left(\psi_+(0) + \int_0^\infty e^{i(x_2-y_2)\xi_1\tau} \psi'_+(\tau) d\tau \right).$$

¹⁶In this situation, $\mathbf{e} = (0, 1)$.

Continuing the successive integration by parts, we arrive to

$$a(x, y, \xi_1) = \sum_{l=0}^{k+1} \frac{1}{[i(y_2 - x_2)]^{l+1}} \psi_+^{(l)}(0) \xi_1^{m-l} + R_+(x, y, \xi_1), \text{ for } \xi_1 > 0,$$

where

$$R_+(x, y, \xi_1) = \frac{1}{[i(y_2 - x_2)]^{k+2}} \xi_1^{m-k-1} \int_0^\infty e^{i(x_2 - y_2) \xi_1 \tau} \psi_+^{(k+2)}(\tau) d\tau.$$

Since $\psi_+^{(l)}(0) = 0$ for all $0 \leq l \leq k-1$, we obtain

$$(35) \quad a(x, y, \xi_1) = \frac{1}{[i(x_2 - y_2)]^{k+1}} \psi_+^{(k)}(0) |\xi_1|^{m-k} \\ + \frac{1}{[i(x_2 - y_2)]^{k+2}} \psi_+^{(k+1)}(0) |\xi_1|^{m-k-1} + R_+(x, y, \xi_1), \quad \xi_1 > 0.$$

Using the same integration by parts technique as above, one can easily show that the function

$$r_+(x, y, \xi_1) = \int_0^\infty e^{i(x_2 - y_2) \xi_1 \tau} \psi_+^{(k+2)}(\tau) d\tau$$

satisfies $r_+(x, y, \xi_1) \in S^0(\mathcal{O} \times \mathbb{R}_+)$ for large $|\xi_1|$. Therefore,

$$R_+(x, y, \xi_1) \in S^{m-k-1}(\mathcal{O} \times \mathbb{R}_+), \text{ for large } |\xi_1|.$$

Similarly

$$(36) \quad a(x, y, \xi_1) = \frac{1}{[i(x_2 - y_2)]^{k+1}} \psi_-^{(k)}(0) |\xi_1|^{m-k} \\ + \frac{1}{[i(x_2 - y_2)]^{k+2}} \psi_-^{(k+1)}(0) |\xi_1|^{m-k-1} + R_-(x, y, \xi_1), \quad \xi_1 < 0.$$

where $\psi_-(\tau) = (1 + \tau^2)^{m/2} \varphi_-(\tau)$ and

$$R_-(x, y, \xi_1) \in S^{m-k-1}(\mathcal{O} \times \mathbb{R}_+), \text{ for large } |\xi_1|.$$

Therefore, from (35) and (36), $a(x, y, \xi_1) \in S^{m-k}(\mathcal{O} \times \mathbb{R})$ for large $|\xi_1|$, with the top order term

$$a_{m-k}(x, y, \xi_1) = \begin{cases} \frac{1}{[i(y_2 - x_2)]^{k+1}} \psi_+^{(k)}(0) \xi_1^{m-k}, & \xi_1 > 0, \\ \frac{1}{[i(y_2 - x_2)]^{k+1}} \psi_-^{(k)}(0) |\xi_1|^{m-k}, & \xi_1 < 0. \end{cases}$$

From the definition of ψ_\pm , it is easy to show that a_{m-k} satisfies (34). This finishes our proof. \square

We now consider a generalization of μ_\pm . Namely, let $\mathbf{e} \in \mathbb{R}^2$ be a unit vector and let us consider the distributions

$$(37) \quad \mu_{\pm \mathbf{e}}(x, y) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i(x-y) \cdot \xi} |\xi|^m \rho_{\mathbf{e}}(\xi) \mathbb{H}(\pm \mathbf{e}^\perp \cdot \xi) d\xi.$$

Let

$$\mathcal{C}_{\mathbf{e}} = \{(x, s \mathbf{e}; x + t \mathbf{e}^\perp, s \mathbf{e}) : x \in \mathbb{R}^2, t, s \in \mathbb{R}, \gamma \neq 0\}.$$

Proposition 4.2. *We have*

$$WF(\mu_{\pm\mathbf{e}})' \subset \Delta \cup \mathcal{C}_{\mathbf{e}}.$$

a) *Near $\Delta \setminus \mathcal{C}_{\mathbf{e}}$, $\mu_{\pm\mathbf{e}}$ is in the space $I^m(\Delta)$ with the symbol:*

$$\sigma(x, \xi) \sim |\xi|^m \rho_{\mathbf{e}}(\xi) \mathbb{H}(\pm \mathbf{e}^\perp \cdot \xi).$$

b) *Assume that $\rho_{\mathbf{e}}$ vanishes to order k on the line $\mathbf{e}^\perp \cdot \xi = 0$. Then, near $\mathcal{C}_{\mathbf{e}} \setminus \Delta$, $\mu_{\pm\mathbf{e}}$ is microlocally in the space $I^{m-k-1/2}(\mathcal{C}_{\mathbf{e}})$. Moreover, given the phase function $\phi(x, y, \tau) = (x - y) \cdot \mathbf{e} \tau$, its principal symbol is given by*

$$\sigma_0(x, x + t \mathbf{e}^\perp, \tau) = \begin{cases} \pm \frac{1}{\sqrt{2\pi}} \frac{1}{(it)^{k+1}} \varphi_{\pm}^{(k)}(0) \tau^{m-k}, & \tau > 0, \\ \pm \frac{1}{\sqrt{2\pi}} \frac{1}{(it)^{k+1}} \varphi_{\pm}^{(k)}(0) |\tau|^{m-k}, & \tau < 0. \end{cases}$$

Here, $\varphi_{\pm} : \mathbb{R} \rightarrow \mathbb{R}$ satisfies

$$\varphi_{\pm}\left(\frac{\xi \cdot \mathbf{e}^\perp}{|\xi \cdot \mathbf{e}|}\right) = \rho_{\mathbf{e}}(\xi), \quad \text{for all } \xi \text{ such that } \pm \xi \cdot \mathbf{e} > 0.$$

Proof. The Proposition 4.2 can be obtained from Propositions 4.1 by a simple change of variables. Indeed, let

$$\eta = (\mathbf{e} \cdot \xi, \mathbf{e}^\perp \cdot \xi), \quad x' = (\mathbf{e} \cdot x, \mathbf{e}^\perp \cdot x), \quad y' = (\mathbf{e} \cdot y, \mathbf{e}^\perp \cdot y).$$

and ρ, μ_{\pm} be defined by

$$\mu_{\pm\mathbf{e}}(x, y) = \mu_{\pm}(x', y'), \quad \rho_{\mathbf{e}}(\xi) = \rho(\eta).$$

By changing the variables in (37), we obtain

$$\mu_{\pm}(x', y') = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i(x'-y') \cdot \eta} |\eta|^m \rho(\eta) \mathbb{H}(\pm \eta_2) d\eta.$$

Applying Propositions 4.1 for μ_{\pm} and translating the result back to $\mu_{\pm\mathbf{e}}$, we finish the proof. \square

5. PROOF OF THEOREM 3.1

Let us first divide the boundary of W_Φ into four rays. Namely, let

$$\mathbf{e}_1 = -\mathbf{e}_3 = (\cos \Phi, \sin \Phi), \quad \mathbf{e}_2 = -\mathbf{e}_4 = (\cos \Phi, -\sin \Phi).$$

and for $j = 1, \dots, 4$:

$$R_j = \{\xi : \xi = r \mathbf{e}_j, r > 0\}.$$

It is obvious that

$$R_1 \cup R_2 \cup R_3 \cup R_4 = \partial W_\Phi \setminus \{0\}.$$

For $j = 1, \dots, 4$, let $\rho_j \in C^\infty(\mathbb{R}^2 \setminus 0)$ be homogeneous of degree zero such that $\rho_j = 1$ in a (small) conic neighborhood of R_j . Moreover, ρ_j is supported inside a small conic neighborhood of R_j ¹⁷.

¹⁷This, in particular, implies $\text{supp}(\rho_j) \cap \text{supp}(\rho_k) = \{0\}$, for $j \neq k$.

We can write:

$$\begin{aligned}
\mu_m(x, y) &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i(x-y) \cdot \xi} |\xi|^m \kappa(\xi/|\xi|) \chi_\Phi(\xi) d\xi \\
&= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i(x-y) \cdot \xi} |\xi|^m \left[1 - \sum_{j=1}^4 \rho_j(\xi) \right] \kappa(\xi/|\xi|) \chi_\Phi(\xi) d\xi \\
&\quad + \sum_{j=1}^4 \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i(x-y) \cdot \xi} |\xi|^m \rho_j(\xi) \kappa(\xi/|\xi|) \chi_\Phi(\xi) d\xi \\
&= \mu^0(x, y) + \sum_{j=1}^4 \mu^j(x, y).
\end{aligned}$$

Properties of μ^0 : We notice that the function

$$b(x, \xi) = |\xi|^m \left[1 - \sum_{j=1}^4 \rho_j(\xi) \right] \kappa(\xi/|\xi|) \chi_\Phi(\xi)$$

is smooth in $\mathbb{R}^2 \times (\mathbb{R}^2 \setminus 0)$. It is, hence, clear that $b(x, \xi)$ satisfies the conditions in Lemma 2.7. Therefore, $\mu^0 \in I^m(\Delta)$ with the symbol $\sigma^0(x, \xi) \sim b(x, \xi)$. This, in particular, implies

$$WF(\mu^0)' \subset \Delta.$$

Properties of μ^1 : Since ρ_1 is supported in a small conic neighborhood of R_1 , we have

$$\chi_\Phi(\xi) = \mathbb{H}(-\mathbf{e}_1^\perp \cdot \xi), \quad \text{for all } \xi \in \text{supp}(\rho_1).$$

Therefore,

$$\begin{aligned}
\mu^1(x, y) &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i(x-y) \cdot \xi} |\xi|^m \kappa(\xi/|\xi|) \rho_1(\xi) \chi_\Phi(\xi) d\xi \\
&= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i(x-y) \cdot \xi} |\xi|^m \kappa(\xi/|\xi|) \rho_1(\xi) \mathbb{H}(-\mathbf{e}_1^\perp \cdot \xi) d\xi.
\end{aligned}$$

Applying Proposition 4.2, we obtain

$$WF(\mu^1)' \subset \Delta \cup \mathcal{C}_1.$$

Moreover, due to Proposition 4.2 a), near $\Delta \setminus \mathcal{C}_1$, μ^1 is microlocally in the space $I^m(\Delta)$ with the full symbol

$$\sigma^1(x, \xi) \sim |\xi|^m \kappa(\xi/|\xi|) \rho_1(\xi) \chi_\Phi(\xi).$$

On the other hand, due to Proposition 4.2 b), near $\mathcal{C}_1 \setminus \Delta$, μ^1 is microlocally in the space $I^{m-k-\frac{1}{2}}(\mathcal{C}_1)$ with the principal symbol

$$(38) \quad \sigma_0^1(x, x + t \mathbf{e}_1^\perp, \tau) = \begin{cases} -\frac{1}{\sqrt{2\pi}} \frac{1}{(it)^{k+1}} \varphi_+^{(k)}(0) \tau^{m-k}, & \tau > 0, \\ -\frac{1}{\sqrt{2\pi}} \frac{1}{(it)^{k+1}} \varphi_-^{(k)}(0) |\tau|^{m-k}, & \tau < 0. \end{cases}$$

given the phase function $\phi_1(x, y, \tau) = (x - y) \cdot \mathbf{e}_1 \tau$. Here, φ_+ is defined by

$$\varphi_\pm\left(\frac{\mathbf{e}_1^\perp \cdot \xi}{|\mathbf{e}_1 \cdot \xi|}\right) = \kappa\left(\frac{\xi}{|\xi|}\right) \rho_1(\xi), \quad \text{for all } \pm \mathbf{e}_1 \cdot \xi > 0.$$

Direct calculations show that

$$\varphi_-^{(k)}(0) = 0, \quad \varphi_+^{(k)}(0) = \varkappa^{(k)}(\Phi),$$

where, we recall, \varkappa is defined by $\varkappa(\phi) = \kappa(\cos \phi, \sin \phi)$. Therefore,

$$\sigma_0^1(x, x + t \mathbf{e}_1^\perp, \tau) = \begin{cases} -\frac{1}{\sqrt{2\pi}} \frac{\varkappa^{(k)}(\Phi)}{(it)^{k+1}} \tau^{m-k}, & \tau > 0, \\ 0. & \end{cases}$$

Properties of μ^3 : Similarly to μ^1 , we obtain

$$WF(\mu^3)' \subset \Delta \cup \mathcal{C}_1.$$

Moreover, near $\Delta \setminus \mathcal{C}_1$, μ^3 is microlocally in the space $I^m(\Delta)$ with the full symbol

$$\sigma^3(x, \xi) \sim |\xi|^m \kappa(\xi/|\xi|) \rho_3(\xi) \chi_\Phi(\xi).$$

On the other hand, near $\mathcal{C}_1 \setminus \Delta$, μ^3 is microlocally in the space $I^{m-k-\frac{1}{2}}(\mathcal{C}_1)$ with the principal symbol

$$\sigma_0^3(x, x + t \mathbf{e}_1^\perp, \tau) = \begin{cases} 0, \\ \frac{1}{\sqrt{2\pi}} \frac{(-1)^k \varkappa^{(k)}(\Phi)}{(it)^{k+1}} |\tau|^{m-k}, & \tau > 0, \end{cases}$$

given the phase function $\phi_1(x, y, \tau) = (x - y) \cdot \mathbf{e}_1 \tau$.

Properties of $\mu^{1,3} = \mu^1 + \mu^3$: Adding up the above results for μ^1 and μ^3 , we obtain for $\mu^{1,3} = \mu^1 + \mu^3$:

$$WF(\mu^{1,3})' \subset \Delta \cup \mathcal{C}_1,$$

Moreover,

i) Near $\Delta \setminus \mathcal{C}_1$, $\mu^{1,3}$ is microlocally in the space $I^m(\Delta)$ with the full symbol

$$\sigma^{1,3}(x, \xi) \sim |\xi|^m \kappa(\xi/|\xi|) [\rho_1(\xi) + \rho_3(\xi)] \chi_\Phi(\xi).$$

ii) Near $\mathcal{C}_1 \setminus \Delta$, $\mu^{1,3}$ is microlocally in the space $I^{m-k-\frac{1}{2}}(\mathcal{C}_1)$ with the principal symbol

$$\sigma_0(x, x + t \mathbf{e}_1^\perp, \tau) = -\frac{1}{\sqrt{2\pi}} \frac{\varkappa^{(k)}(\Phi)}{[\operatorname{sgn}(\tau) i t]^{k+1}} |\tau|^{m-k},$$

given the phase function $\phi_1(x, y, \tau) = (x - y) \cdot \mathbf{e}_1 \tau$.

Properties of $\mu^{2,4} = \mu^2 + \mu^4$: Similarly to $\mu^{1,3}$, we obtain for $\mu^{2,4} = \mu^2 + \mu^4$:

$$WF(\mu^{2,4})' \subset \Delta \cup \mathcal{C}_2.$$

Moreover,

i) Near $\Delta \setminus \mathcal{C}_2$, $\mu^{2,4}$ is microlocally in the space $I^m(\Delta)$ with the full symbol

$$\sigma^{2,4}(x, \xi) \sim |\xi|^m \kappa(\xi/|\xi|) [\rho_2(\xi) + \rho_4(\xi)] \chi_\Phi(\xi).$$

ii) Near $\mathcal{C}_2 \setminus \Delta$, $\mu^{2,4}$ is microlocally in the space $I^{m-k-\frac{1}{2}}(\mathcal{C}_2)$ with the principal symbol

$$\sigma_0(x, x + t \mathbf{e}_2^\perp, \tau) = \frac{1}{\sqrt{2\pi}} \frac{\varkappa^{(k)}(-\Phi)}{[i \operatorname{sgn}(\tau) t]^{k+1}} |\tau|^{m-k},$$

given the phase function $\phi_2(x, y, \tau) = (x - y) \cdot \mathbf{e}_2 \tau$.

Properties of μ_m : Adding up all the above characterizations of $\mu^0, \mu^{1,3}, \mu^{2,4}$, we obtain Theorem 3.1 a) & b), and the inclusion¹⁸

$$WF(\mu_m)' \subset \Delta \cup \mathcal{C}_1 \cup \mathcal{C}_2.$$

Since $\sigma(x, \xi) \sim 0$ if $\xi \notin \text{cl}(W_\Phi)$, we arrive to (see, e.g., [Hör71, Proposition 2.5.7]):

$$WF(\mu) \subset \Delta_\Phi \cup \mathcal{C}_1 \cup \mathcal{C}_2.$$

This finishes our proof.

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¹⁸Here, we have used the inclusion $WF(\mu^1 + \mu^2) \subset WF(\mu^1) \cup WF(\mu^2)$.

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